Convergence of sequences

(The following problem is borrowed from page 599 of *Calculus* (8th ed.) by Larson, Hostetler, and Edwards.)

Problem:

Find a definition for the sequence $\{a_n\}$ whose first five terms are

$$-\frac{2}{1}, \frac{8}{2}, -\frac{26}{6}, \frac{80}{24}, -\frac{242}{120}, \ldots$$

and determine whether it converges or diverges.

Solution:

By inspection,

$$a_n = (-1)^n \frac{3^n - 1}{n!}$$

A theorem states that if the absolute ratio of successive terms is less than 1 for large n, then the sequence will converge to 0, i.e.

$$\text{if} \quad \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| 1, \text{ then} \quad \lim_{n\to\infty} a_n = 0$$

(Note that this is similar to, but different from, the ratio test for convergence of *series*.)

In our example,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} \frac{3^{n+1}-1}{(n+1)!}}{(-1)^n \frac{3^n-1}{n!}} \right|$$

We can simply ignore the $(-1)^n$ factors, because we are only interested in the ratio of the absolute values. Simplifying, we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} \frac{3^{n+1} - 1}{3^n - 1} = \lim_{n\to\infty} \frac{1}{n+1} \frac{3^{n+1} - 1}{3^n - 1}$$

As $n \to \infty$, the first fraction in this limit goes to 0, while the second goes to 3. Since the limit of a product is the same as the product of the limits, the overall limit is 0, and by the theorem stated above, the sequence converges to 0, i.e.

$$\lim_{n \to \infty} a_n = 0$$